

Melvin solution with a dilaton potential

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Abstract

We find new Melvin-like solutions in Einstein-Maxwell-dilaton gravity with a Liouville-type dilaton potential. The properties of the corresponding solution in Freedman-Schwarz gauged supergravity model are extensively studied. We show that this configuration is regular and geodesically complete but do not preserve any supersymmetry. An exact solution describing travelling waves in this Melvin-type background is also presented.

1 Introduction

The Melvin magnetic universe is a regular and static, cylindrically symmetric solution to Einstein-Maxwell theory describing a bundle of magnetic flux lines in gravitational-magnetostatic equilibrium [1]. This solution has a number of interesting features, providing the closest approximation in general relativity for an uniform magnetic field. The nonsingular nature of this solution (at the cost of losing the asymptotic flatness) motivated Melvin to appoint his solution as a magnetic "geon". There exist a fairly extensive literature on the properties of this magnetic universe, including generalizations in several directions. Rotating and time-dependent magnetic universes have been presented in [2], as well as gravitational waves travelling in a magnetic universe [3]. Of particular interest are black hole solutions in universes which are asymptotically Melvin [4, 5]. Recently multidimensional generalizations of the Melvin magnetic universe attracted much attention as fluxbranes of the superstring theory. The Melvin solution has been generalized also for a gravity theory minimally coupled to any nonlinear electromagnetic theory, including Born-Infeld theory [6].

Most of the work on this subject, including pair creation of charged black holes in a Melvin universe background [7, 8] has been done in a Einstein-Maxwell theory and in a generalization of this theory which include a dilaton ϕ , whose action is

$$I = \int d^4x \sqrt{-g} \left(\frac{R}{4} - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{4}e^{-2a\phi}F^2 \right), \quad (1)$$

where R is the scalar curvature, $F_{\mu\nu}$ is the Maxwell field. The constant a governs the coupling of ϕ to $F_{\mu\nu}$. For $a = 0$ the scalar field decouples and we have the original Einstein-Maxwell theory with a minimally coupled scalar field, while $a = 1$ is a consistent truncation of the low-energy string theory action. The value $a = \sqrt{3}$ corresponds to the standard Kaluza-Klein theory.

The dilaton Melvin solution for an arbitrary a takes the form [7, 8]

$$ds^2 = \Lambda^{\frac{2}{1+a^2}}(d\rho^2 + dz^2 - dt^2) + \Lambda^{-\frac{2}{1+a^2}}\rho^2 d\varphi^2, \quad (2)$$

$$e^{-2a(\phi-\phi_0)} = \Lambda^{\frac{2a^2}{1+a^2}}, \quad A_\varphi = e^{a\phi_0} \frac{B_0 \rho^2}{2\Lambda},$$

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where

$$\Lambda = 1 + \left(\frac{1+a^2}{4}\right)B_0^2\rho^2.$$

The solution is parametrized by ϕ_0 , the value of the scalar field on the symmetry axis and B_0 , which characterizes the central strength of the magnetic field. Although not asymptotically flat, the geometry of this solution is singularity free and geodesically complete. A curious property of (2) is that the total flux

$$\Phi = \oint_{\infty} A_{\varphi} = e^{a\phi_0} \frac{4\pi}{1+a^2} \frac{1}{B_0} \quad (3)$$

is finite and inversely proportional to B_0 . However, in the limit $B_0 \rightarrow 0$, even if the geometry becomes flat and the field strength goes to zero at the centre, the total flux diverges.

A natural generalization of the action (1) is to include a dilaton potential term $V(\phi)$, which will act such as an effective (position dependent) cosmological constant. Black hole solutions for this case have been considered by many authors, and generally present properties that differ significantly from the standard Einstein-Maxwell-dilaton theory [9].

It is therefore natural to ask whether magnetic universe solutions also exist for a nonzero $V(\phi)$ and how the properties of the solution are affected by the potential term. To our best knowledge, to date this question has not been answered in the literature.

It is the purpose of this paper to approach this problem and to present new Melvin-type exact solution. Given the inherent difficulties involved in studies of models with nontrivial dilaton potentials, the present paper is intended as a first step only and we restrict ourselves to the case of a Liouville-type potential $V(\phi) = V_0 e^{2b\phi}$. Special attention is paid to the case $a = b = 1$ which corresponds to the $N = 4$, $D = 4$ gauged $SU(2) \times SU(2)$ supergravity, known also as the Freedman-Schwarz (FS) model [11]. The corresponding Melvin solution has a particularly simple form in this case, which allows us to discuss its property to some extend.

This paper is organized as follows. In Section 2 we describe the basic formalism, derive the field equations and present explicit solution in several cases. In Section 3 we present the Melvin solution in context of the $N = 4$, $D = 4$ gauged supergravity model solution and analyse its properties. We conclude with Section 4 where the results are compiled.

2 General framework and equations of motion

The field equations are obtained by varying the action (1) (where we included the dilaton potential term $V(\phi)$) with respect to the field variables g_{ij} , ϕ and A_i

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 2\left(\partial_{\mu}\phi\partial_{\nu}\phi - \frac{1}{2}g_{\mu\nu}\partial_{\rho}\phi\partial^{\rho}\phi + V(\phi)g_{\mu\nu} + e^{-2a\phi}(F_{\mu\rho}F_{\nu}{}^{\rho} - \frac{1}{4}g_{\mu\nu}(F_{\rho\sigma}F^{\rho\sigma}))\right), \quad (4)$$

$$\frac{1}{\sqrt{-g}}\partial_{\mu}(\sqrt{-g}\partial^{\mu}\phi) = -\frac{a}{2}e^{-2a\phi}F_{\rho\sigma}F^{\rho\sigma} - \frac{\partial V(\phi)}{\partial\phi}, \quad (5)$$

$$0 = \partial_{\mu}(\sqrt{-g}e^{-2a\phi}F^{\mu\nu}). \quad (6)$$

We start by considering a line element on the form

$$ds^2 = M^2(r)dr^2 + N^2(r)d\varphi^2 + P^2(r)(dz^2 - dt^2). \quad (7)$$

The symmetries of this line element are the same as that of the original Melvin solution (2). There are again four Killing vectors corresponding to translation along the t and z directions, a rotation along the z axis and the $t - z$ boost (note that this assures that (7) is cylindrically symmetric since it admits a G_2 on S_2 group of isometries containing an axial symmetry [10]). One can fix the residual gauge freedom $r \rightarrow \tilde{r}$ of this metric ansatz by imposing a gauge condition on the function M , N , P .

We suppose also that the Maxwell potential presents only on nonvanishing component $A_\varphi(r)$. Thus the Maxwell equations can easily be integrated to obtain

$$F_{r\varphi} = B_0 e^{2a\phi} \frac{MN}{P^2}, \quad (8)$$

where B_0 is a constant of integration. The Einstein and scalar field equations then yields

$$\frac{1}{2} \frac{P'^2}{P^2 M^2} + \frac{N' P'}{N P M^2} = \frac{1}{2} B_0^2 \frac{e^{2a\phi}}{P^4} + \frac{\phi'^2}{2M^2} + V(\phi), \quad (9)$$

$$\frac{1}{2} \frac{P'^2}{P^2 M^2} - \frac{M' P'}{P M^3} + \frac{P''}{M^2 P} = \frac{1}{2} B_0^2 \frac{e^{2a\phi}}{P^4} - \frac{\phi'^2}{2M^2} + V(\phi), \quad (10)$$

$$\frac{1}{2} \frac{M' P'}{P M^3} - \frac{1}{2} \frac{P''}{M^2 P} - \frac{1}{2} \frac{N''}{M^2 N} - \frac{1}{2} \frac{N' P'}{M^2 N P} + \frac{1}{2} \frac{M' N'}{M^3 N} = \frac{1}{2} B_0^2 \frac{e^{2a\phi}}{P^4} + \frac{\phi'^2}{2M^2} - V(\phi), \quad (11)$$

$$\left(\frac{N P^2 \phi'}{M} \right)' = -a B_0^2 e^{2a\phi} \frac{MN}{P^2} - M N P^2 \frac{\partial V(\phi)}{\partial \phi}, \quad (12)$$

where a prime denotes a derivative with respect to r . These equations take a simpler form for the gauge choice

$$M^2 = e^{3u+v}, \quad N^2 = e^{u-v}, \quad P^2 = e^{u+v}, \quad (13)$$

the relation (8) translating to

$$F_{r\varphi} = B_0 e^{2a\phi+u-v}. \quad (14)$$

The Einstein and dilaton equations have the simple form

$$\begin{aligned} u'' &= 4V(\phi) e^{3u+v}, \\ v'' &= 2B_0^2 e^{u-v+2a\phi}, \\ \phi'' &= -a B_0^2 e^{u-v+2a\phi} - e^{3u+v} \frac{\partial V(\phi)}{\partial \phi}, \end{aligned} \quad (15)$$

together with the constraint

$$3u'^2 + 2u'v' - v'^2 - 4\phi'^2 - 8e^{3u+v}V(\phi) - 4B_0^2 e^{u-v+2a\phi} = 0.$$

2.1 Exact solutions

To proceed further, however, we must choose a particular form of $V(\phi)$. In this paper we specialize to the Liouville potential $V = V_0 e^{2b\phi}$, which is simple enough to allow explicit solutions. This form of potential is also of practical interest, since it frequently arises in the bosonic sector of gauged supergravities. Unfortunately, even for this simple form of $V(\phi)$, we were unable to obtain a general solution of the field equations, valid for every a, b, V_0 . However, several special values of these parameters allow for exact solutions.

The scalar field equation takes a particularly simple form in the (u, v) metric parametrization

$$\phi'' = -\frac{1}{2}(av'' + bu''), \quad (16)$$

and can be integrated one time to $\phi' = -1/2(av' + bu') + \text{const.}$ For the choice $\text{const.} = 0$ the equations (15) read

$$\begin{aligned} u'' &= 4V_0 e^{2b\phi_0} e^{(3-b^2)u+(1-ab)v}, \\ v'' &= B_0^2 e^{2a\phi_0} e^{(1-ab)u-(1+a^2)v}, \end{aligned} \quad (17)$$

with $\phi = -1/2(av + bu) + \phi_0$, ϕ_0 being an arbitrary constant. The system (17) admits exact solutions for special values of a, b .

2.1.1 Kaluza-Klein solutions

In the case $a = 1/b = \sqrt{3}$ the action of the theory corresponds to the Kaluza-Klein reduction of the five dimensional gravity with a cosmological constant

$$I = \int d^5x \sqrt{-g_5} \frac{1}{4} (R_5 - 2\Lambda_5), \quad (18)$$

in which the five-metric is parametrized as follows

$$ds_5^2 = e^{-\frac{4\phi}{\sqrt{3}}} (dx^5 + 2A_\mu dx^\mu) + e^{\frac{2\phi}{\sqrt{3}}} g_{ij} dx^i dx^j. \quad (19)$$

The value of the cosmological constant fixed the value V_0 through $V_0 = -2\Lambda_5$. Therefore one may hope to generate Melvin-like dilaton solutions starting with suitable vacuum (anti-)de Sitter five dimensional configurations.

For this choice of (a, b) , an exact solution of the equations (17) reads

$$u = u_0 - \frac{3}{4} \log \sinh(r + c_1), \quad v = v_0 + \frac{1}{2} \log \cosh(r + c_2), \quad \phi = \phi_0 - \frac{\sqrt{3}}{2} (v + \frac{u}{3}), \quad (20)$$

where ϕ_0, c_1, c_2 are arbitrary constants, $e^{-4v_0} = 2B_0^2 e^{2\sqrt{3}\phi_0}$, $e^{8u_0/3} = 3e^{-2\phi_0/\sqrt{3}}/(16V_0)$, while the magnetic field is given by (14).

However, different from the $\Lambda_5 \rightarrow 0$ limit, the solution in this case present some unphysical properties and seems to be less physical relevant. In particular, similar to other cases [12], this solution presents a singular axis of symmetry, since the limit $\lim_{r \rightarrow r_0} g_{\varphi\varphi}/((r - r_0)^2 g_{rr})$ (where r_0 is a zero of the metric function $g_{\varphi\varphi}$) is singular for every choice of c_1, c_2 . Therefore, this is not the type of configuration we are interested.

2.1.2 The case $a = b$

The particular case $a = b$ allows us also to find an exact solution. The gauge choice $M = 1/N$ implies from the Einstein equations (9) and (10) the simple relation $P''/P = -\phi'^2$. We further assume $P(r) = r^n$, which leads to the solution

$$\begin{aligned} N^2(r) &= \frac{1}{M^2(r)} = Cr^{1-2n} + \frac{2V_0 e^{2a\phi_0}}{n(4n-1)} r^{2n} - B_0^2 e^{2a\phi_0} \frac{r^{-2n}}{n}, \quad P(r) = r^n, \\ \phi &= \phi_0 \pm \sqrt{n(1-n)} \log r, \quad F_{r\varphi} = B_0 e^{2a\phi} r^{-2n}, \end{aligned} \quad (21)$$

C is an arbitrary constant of integration. The Einstein equations impose also

$$a = b = \mp \sqrt{(1-n)/n}, \quad (22)$$

which implies that we must restrict to $0 < n < 1$ and $a \neq \sqrt{3}$.

The parameters V_0, C in the expression of N^2 should be restricted such that $N^2(r) > 0$ asymptotically. For $a^2 > 3$, the term r^{1-2n} is dominant for large values of r and we should impose that $C > 0$. For values of $a^2 < 3$, the second term will be dominant as $r \rightarrow \infty$ and the value of V_0 should be positive. We can also proven that this solution presents a singularity as $r \rightarrow 0$ and the function $N(r)$ has at least one zero. Therefore the g_{rr} metric function is negative for $r < r_0$ and positive for $r > r_0$ (r_0 should be considered the first solution of the equation $N(r) = 0$ when coming from infinity). However, when g_{rr} becomes negative so does $g_{\varphi\varphi}$ and this leads to an apparent change of signature from $+2$ to -2 . This indicates that, comparable to similar situations (see *e.g.* [13]), an incorrect extension is being used and one should choose a different continuation to describe the region $r < r_0$. Therefore the physical region of our solution is $r \geq r_0$. The singularity at $r = r_0$ generally corresponds to the origin of the flat space in polar coordinates. In the neighbourhood of $r = r_0$ we introduce a new coordinate ρ defined by

$$\rho = 2\sqrt{\frac{r - r_0}{g'_{\varphi\varphi}(r_0)}}, \quad (23)$$

and the $r - \varphi$ section of the metric (7) reads

$$d\sigma_1^2 \simeq d\rho^2 + (g'_{\varphi\varphi}(r_0)) \rho^2 d\varphi^2. \quad (24)$$

Hence the regularity condition as $r \rightarrow r_0$ imposes to identify the coordinate φ with a period $4\pi r_0 n e^{-2a\phi_0} / (B_0^2 r_0^{-2n} + 2V_0 r_0^{2n})$. As $r \rightarrow r_0$ the $z - t$ sector of the metric (7) is also regular and reads $d\sigma_2^2 \simeq r_0^2 (dz^2 - dt^2)$.

Here we note that, if one insists that the coordinate φ is identified with a period 2π , this metric has the property of a cosmic string as is found in the Abelian Higgs model (see *e.g.* [14]). There will appear an angle deficit (flat space minus a wedge), if $r_0 n e^{-2a\phi_0} / (B_0^2 r_0^{-2n} + 2V_0 r_0^{2n}) \equiv G\mu < \frac{1}{2}$, where μ is the linear mass density of the string and G the gravitation constant. In general, it would be of interest to compare this model with the Higgs model, where the magnetic flux (see eqs. (3) and (29)) is quantized and depends on the number of zero's of the Higgs field.

However, the general properties of these solutions are rather difficult to discuss for a generic a (in particular we cannot write a form of the line element similar to (2)). Therefore we prefer to focus for the rest of this paper on the particular case $a = b = 1$.

We close this Section by remarking that the solution (21) and a class of topological black hole solutions in Einstein-Maxwell theory with a Liouville-type dilaton potential [15] shares the same Euclidean section and are related by an analytical continuation. The double "Wick rotation" $t \rightarrow iy$, $\varphi \rightarrow i\tau$, together with $B_0 \rightarrow iQ$ in (7) leads to the solution

$$\begin{aligned} ds^2 &= \frac{dr^2}{F(r)} + r^n (dy^2 + dz^2) - F(r) d\tau^2, \\ \phi &= \phi_0 \pm \sqrt{n(1-n)} \log r, \quad F_{r\tau} = Q e^{2a\phi} r^{-2n}, \\ F(r) &= C r^{1-2n} + \frac{2V_0 e^{2a\phi_0}}{n(4n-1)} r^{2n} + Q^2 e^{2a\phi_0} \frac{r^{-2n}}{n}, \end{aligned} \quad (25)$$

(where again $a = b = \mp \sqrt{(1-n)/n}$), corresponding to a topological black hole configuration, whose properties are discussed in [15]. Here it may be useful to remark that the all known exact solutions of the Einstein-Maxwell-dilaton theory with a dilaton potential have been obtained within a restricted ansatz. Different from the asymptotically flat case [16], the question of solution's uniqueness is not answered in the presence of a dilaton potential. Thus, other black hole solutions are likely to exist apart from (25), which, after a suitable analytical continuation, will lead to new Melvin-type configurations.

3 Melvin solution in Freedman-Schwarz model

For $a = b = 1$, the action (26) (with the gauge potential term $V(\phi) = V_0 e^{2\phi}$ included) corresponds to a consistent truncation of the bosonic sector of the $N = 4$, $D = 4$ gauged $SU(2) \times SU(2)$ supergravity known also as the Freedman-Schwarz model [11].

The action of the FS model includes a vierbein e_μ^m , four Majorana spin-3/2 fields ψ_μ^I , vector and six vector fields A_μ^a and B_μ^a ($a = 1, 2, 2$) with independent gauge coupling constants g_1 and g_2 , respectively, four Majorana spin-1/2 fields χ^I , the axion η and the dilaton ϕ [11]. The bosonic part of the action reads

$$S = \int d^4x \sqrt{-g} \left(\frac{1}{4} R - \frac{1}{2} (\partial_\mu \phi \partial^\mu \phi + e^{4\phi} \partial_\mu \eta \partial^\mu \eta) - \frac{1}{4} e^{-2\phi} (F_{\mu\nu}^a F^{a\mu\nu} + G_{\mu\nu}^a G^{a\mu\nu}) + \frac{\lambda^2}{8} e^{2\phi} \right), \quad (26)$$

where $F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g_1 \epsilon_{abc} A_\mu^b A_\nu^c$, $G_{\mu\nu}^a = \partial_\mu B_\nu^a - \partial_\nu B_\mu^a + g_2 \epsilon_{abc} B_\mu^b B_\nu^c$. The gauge coupling constants of the theory fix the value of $V_0 = \lambda^2/8$ (with $\lambda^2 = g_1^2 + g_2^2$, thus $V(\phi)$ corresponding to a negative effective cosmological constant).

A review of the known exact solutions of this theory should start with stable electrovac configuration found by Freedman and Gibbons [17], which is a product manifold $AdS_2 \times R^2$, and preserves one quarter or one half of the supersymmetries, the latter case occurring if one of the two gauge coupling constants

vanishes. There are also other supersymmetric vacua of the FS model, in particular the domain wall solution [18, 20] preserving also one half of the supersymmetries. This solution has vanishing gauge fields and is purely dilatonic. Furthermore, BPS configurations involving a non-zero axion were found by Singh [19, 20]. Abelian BPS black hole solutions with toroidal event horizon were constructed in [21]. The $N = 4$, $D = 4$ gauged $SU(2) \times SU(2)$ supergravity has also the important property to allow for exact solutions with nonabelian matter fields as discovered by Chamseddine and Volkov [22, 23] (see also [26]). It was shown recently that the FS model can be obtained by compactifying $N = 1$ ten dimensional supergravity on the $SU(2) \times SU(2)$ group manifold [22, 24] (previously also, a Kaluza-Klein interpretation was given in [25]). Therefore it may be worthwhile to look for new exact solutions of FS model, keeping in mind the recent developments in massive supergravities.

Here we break the gauge group $SU(2) \times SU(2)$ to $U(1) \times U(1)$, considering the Abelian reduction of the theory (i.e. $A_\mu^a = A_\mu \delta^{a3}$, $B_\mu^a = B_\mu \delta^{a3}$). For purely magnetic or electric gauge field ansatz, $\eta = 0$ is a consistent truncation of the FS model.

Since we'll be later interested in the supersymmetric properties of the Melvin-like solutions, we present here the general configuration with two electromagnetic fields (the one field limit is obvious). The equations of motion (4)-(6) can easily be generalized to this case, the only difference being the occurrence of the corresponding $G_{\mu\nu}$ terms (the second electromagnetic field $G_{\mu\nu}$ satisfies an equation similar to (6)).

In the case $a = b = 1$, the field equations admit the simple solution

$$\begin{aligned} ds^2 &= \Lambda(d\rho^2 + dz^2 - dt^2) + \frac{\sinh^2(c\lambda\rho)}{\lambda^2 c^2 \Lambda} d\varphi^2, \\ \phi &= -\frac{\log \Lambda(\rho)}{2} + \log c + \frac{1}{2} \log 2, \\ A_\mu &= \delta_{\mu\varphi} \frac{2B_0 \cos u}{\lambda} \frac{\sinh^2(c\lambda\rho/2)}{\Lambda}, \quad B_\mu = \delta_{\mu\varphi} \frac{2B_0 \sin u}{\lambda} \frac{\sinh^2(c\lambda\rho/2)}{\Lambda}, \\ \text{with } \Lambda &= \frac{\alpha}{\lambda} \left(1 + (B_0^2 + 1) \sinh^2\left(\frac{c\lambda\rho}{2}\right) \right), \end{aligned} \tag{27}$$

where the potential vectors are given in a regular gauge. Here α, u, c are arbitrary real constants. This solution presents a number of similarities with the famous Melvin solution, which is a solution of $D = 4$, $N = 2$ supergravity. As expected, in the limit $\lambda \rightarrow 0$ (taken together with the rescaling $B_0 \rightarrow B_0 \sqrt{2}/(\lambda c)$) the above solution reduces to the $a = 1$ Melvin solution in (2).

3.1 Solution properties

This spacetime has no horizons and no coordinate singularities except from $\rho = 0$. If we suppose an angular coordinate range $0 \leq \varphi < 2\pi$ and impose the usual regularity condition on the symmetry axis

$$\lim_{\rho \rightarrow 0} \frac{1}{\rho^2} \frac{g_{\varphi\varphi}}{g_{\rho\rho}} = 1, \tag{28}$$

we find $\alpha = \lambda$. One can also verify that the ratio α/λ has no physical meaning, since it can be absorbed in a rescaling of the coordinates x^i (for a different periodicity of φ , a simple rescaling of the angular coordinate allows (28) to be satisfied). Thus, similar to the original dilaton Melvin case, the regular solution possesses only two essential parameters c and B_0 , related to the initial value of the scalar field and the magnitude of the magnetic field.

The flux of the magnetic field $\Phi = \int (F + G) = \oint_{r \rightarrow \infty} (A_\varphi + B_\varphi)$ is also finite

$$\Phi = \frac{4\pi B_0 (\cos u + \sin u)}{\lambda(B_0^2 + 1)}, \tag{29}$$

and, different from the case without a dilaton potential, vanishes in the limit $B_0 \rightarrow 0$. This is a manifestation of the different asymptotic structure of spacetime, since these solutions are not asymptotically flat nor (anti-)de Sitter.

It is easy to check the absence of singularities in scalar polynomials of the curvature. We find

$$R = -\frac{c^2\lambda^2}{16\Lambda^3} \left(13 - 6B_0^2 + 13B_0^4 - 16(B_0^4 - 1) \cosh(c\lambda\rho) + 3(B_0^2 + 1) \cosh(2c\lambda\rho) \right), \quad (30)$$

for the curvature scalar, while the explicit expressions for $R_{\mu\nu}R^{\mu\nu}$ and the Kretschmann scalar are very long and will not be given here. However, they are both on the form $w(r)/\Lambda^6$, where $w(r)$ is an everywhere finite function.

We can show also that the spacetime described by (27) is both null and timelike geodesically complete. For the general line element (27), the equations of the geodesics have the four straightforward first integrals

$$\begin{aligned} P_z &= \Lambda \dot{z}, \\ L &= \frac{\sinh^2(c\lambda\rho)}{\lambda^2 c^2 \Lambda} \dot{\varphi}, \\ E &= \Lambda \dot{t}, \\ -\varepsilon &= \Lambda(\dot{\rho}^2 + \dot{z}^2 - \dot{t}^2) + \frac{\sinh^2(c\lambda\rho)}{\lambda^2 c^2 \Lambda} \dot{\varphi}^2, \end{aligned} \quad (31)$$

where a superposed dot stands for as derivative with respect to the parameter τ and $\varepsilon = 1$ or 0 for timelike or null geodesics respectively. τ is an affine parameter along the geodesics; for timelike geodesics, τ is the proper time. From the above equations we get the differential equation for the radial coordinate

$$\left(\frac{d\rho}{d\tau} \right)^2 = \frac{E^2 - P_z^2}{\Lambda^2} - \frac{\varepsilon}{\Lambda} - \frac{\lambda^2 c^2 L^2}{\sinh^2(c\lambda\rho)}. \quad (32)$$

Unfortunately, different from the original Melvin solution, this equation cannot be solved in closed form (except for $\varepsilon = 0$ which leads to a very complicated expression). However, some general qualitative properties of the geodesic motion can easily be deduced. From (31) and (32) we find

$$\frac{d\rho}{dt} = \left(1 - \frac{P_z^2}{E^2} - \frac{\varepsilon\Lambda}{E^2} - \frac{\lambda^2 c^2 L^2}{E^2} \frac{\Lambda^2}{\sinh^2(c\lambda\rho)} \right)^{1/2}. \quad (33)$$

It is evident that, since the term Λ/E^2 in the above relation diverges asymptotically while $\Lambda^2/\sinh^2(c\lambda\rho)$ tends to a constant value in the same limit, any massive particle cannot escape to radial infinity (a similar situation was found in the original Melvin universe [27]). The limit $r \rightarrow \infty$ is allowed only for massless particle satisfying the condition $1 - P_z^2/E^2 - \lambda^2 c^2 L^2 (B_0^2 + 1)^2 / (4E^2) > 0$. Also, $\rho = 0$ is not an admissible value for particles with nonvanishing angular momentum L .

The geodesics of constant (ρ, z) are given by

$$\varphi = \pm \frac{c\lambda\Lambda}{2\sinh(c\lambda\rho)} \left(\frac{B_0^2 + 1}{\cosh^4(c\lambda\rho/2) + B_0^2 \sinh^4(c\lambda\rho/2)} \right)^{1/2} t, \quad (34)$$

and are circles about the symmetry axis. In this case, different from the Melvin universe, a free particle can move in a circular orbit for any finite values of the radial coordinate.

At large ρ , the spacetime (7) approaches a conformally flat geometry

$$ds^2 \simeq \frac{\alpha(B_0^2 + 1)}{4\lambda} e^{c\lambda\rho} (d\rho^2 + dz^2 + dw^2 - dt^2), \quad (35)$$

where $w = 2\varphi/((B_0^2 + 1)c\alpha)$ is a compact coordinate. Thus, our generalized Melvin universe does not approach one of the dilaton Melvin solutions (2), which is clearly a consequence of the presence of a dilaton potential with no fixed points in the action.

3.2 A generation procedure

We can easily prove the following property. Let $(g_{\mu\nu}, A_\mu, \phi)$ be an axisymmetric solution of the field equations (4)-(6), for a Lioville-type potential $V = V_0 e^{2b\phi}$ with $b = 1/a$. All the fields are independent of the azimuthal coordinate φ . Let the other three coordinates be denoted by $\{x^i\}$. Suppose also that $A_i = g_{i\varphi} = 0$. Then a new solution of the equations of motion is given by

$$\begin{aligned} g'_{ij} &= \Lambda^{\frac{2}{1+a^2}} g_{ij}, \quad g'_{\varphi\varphi} = \Lambda^{-\frac{2}{1+a^2}} g_{\varphi\varphi}, \quad e^{-2a\phi'} = e^{-2a\phi} \Lambda^{\frac{2a^2}{1+a^2}}, \\ A'_\varphi &= -\frac{2}{(1+a^2)B\Lambda} \left(1 + \frac{(1+a^2)}{2} B A_\varphi\right), \quad \Lambda = \left(1 + \frac{(1+a^2)}{2} B A_\varphi\right)^2 + \frac{(1+a^2)B^2}{4} g_{\varphi\varphi} e^{2a\phi}. \end{aligned} \quad (36)$$

As explicitly proven in [8, 28], the action principle (1) is invariant under the above transformation, for any values of a . However, we can easily prove that $\sqrt{-g'} e^{2\phi'/a} = \sqrt{-g} e^{2\phi/a}$, and thus the supplementary Lioville potential term in the total action $\int d^4x \sqrt{-g} V_0 e^{2b\phi}$ preserves also the form for $b = 1/a$.

The dilaton Melvin metric (2) is generated applying this procedure to the Minkowski spacetime, which is not a vacuum of the theory for $V(\phi) \neq 0$. However, in our case we can easily generate magnetic configurations starting with vacuum dilaton solutions. For example, the regular Melvin solution of FS model (7) (with $\alpha = \lambda$) is found by using the pure dilaton seed metric ($B_0 = 0$)

$$\begin{aligned} ds^2 &= \cosh^2\left(\frac{c\lambda\rho}{2}\right)(d\rho^2 + dz^2 - dt^2) + \frac{\sinh^2(c\lambda\rho/2)}{\lambda^2 c^2} d\varphi^2, \\ \phi &= -\log(\cosh(c\lambda\rho/2)) + \log c\sqrt{2}. \end{aligned} \quad (37)$$

Other magnetic solutions are found by using different seed metrics known in the literature.

One may hope to generate in this way more complex configurations, black hole solutions immersed in a magnetic universe being of special interest. For $\phi = V(\phi) = 0$, the seed metric is the Schwarzschild one and the corresponding solution was constructed about thirty years ago by Ernst [5].

However, all known black hole solutions of the FS model have been found for an ansatz satisfying $g_{\varphi\varphi} = e^{-2\phi}$. Therefore the transformation (36) does not lead to new solutions in this case. The situation is different for $1/b = a \neq 1$, where this generation procedure leads to nontrivial black hole solutions in a magnetic universe background. We hope to come back on this point in the future.

3.3 Travelling waves in the magnetic universe

A travelling wave is a wave that propagates without any change in amplitude or shape, while the induced perturbation does not need to be small. Similar to the original Melvin universe and its nonlinear electrodynamics version [6], the Melvin-type solution (27) allows for travelling waves generalizations.

To find this solution, we follow the general approach presented in [2], and consider a generalized Kerr-Schild metric ansatz

$$\bar{g}_{\mu\nu} = g_{\mu\nu} + \Lambda^{-2} \Psi(u, \rho, \varphi) k_\mu k_\nu, \quad (38)$$

where $g_{\mu\nu}, \Lambda$ are those of the background Melvin-type metric (27), $\Psi(u, \rho, \varphi)$ is a scalar whose form is determined by the field equations, while $k_\mu = \delta_\mu^u$. Here we have introduced the null coordinates $(u, v) = (z \pm t)/\sqrt{2}$, k_μ being a null Killing vector.

Similar to the cases discussed by Garfinkle and Melvin [2] or Gibbons and Herdeiro [6], the field equations are solved if the function $\Psi(u, \rho, \varphi)$ is harmonic in the unperturbed Melvin-type universe (27), *i.e.* $\nabla^2 \Psi(u, \rho, \varphi) = 0$, where ∇^2 is considered with respect to $g_{\mu\nu}$. In the new travelling wave solution, the electromagnetic and dilaton field are still given by (27). By separating variables we find that $\Psi(u, \rho, \varphi) = f(u)P(\rho) \cos \nu(\varphi - \varphi_0)$, where $f(u)$ is an arbitrary smooth function giving the profile of the wave (for $f(u) = 0$ we find the background metric (27)).

The function $P(\rho)$ is a solution of the equation

$$\sinh(c\lambda\rho) \frac{d}{d\rho} \left(\sinh(c\lambda\rho) \frac{dP(\rho)}{d\rho} \right) - \frac{c^2 \lambda^2 \nu^2}{4} \left(1 - B_0^2 + (1 + B_0^2) \cosh^2(c\lambda\rho) \right)^2 P(\rho) = 0, \quad (39)$$

(a simpler form of this equation is obtained by taking $\cosh(c\lambda\rho) = x$). The geometry describing travelling waves in a Melvin model with Liouville dilaton potential is then

$$ds^2 = \Lambda(\rho) \left(d\rho^2 + 2dudv + f(u) \cos \nu (\varphi - \varphi_0) P(\rho) du^2 \right) + \frac{\sinh^2(c\lambda\rho)}{\lambda^2 c^2 \Lambda} d\varphi^2. \quad (40)$$

Unfortunately, the equation (39) can be solved in closed form for special values of (ν, B_0) only. The solution with $\nu = 0$ is $P(\rho) = c_1 \log(\tanh(c\lambda\rho/2)) + c_2$, while for $B_0 = 1$, $\nu = 1/2$ we find $P(\rho) = (c_3 \sinh(c\lambda\rho/\sqrt{2}) + c_4 \cosh(c\lambda\rho/\sqrt{2}))/\sqrt{\sinh(c\lambda\rho)}$, with c_i arbitrary real constants. We remark that these configurations presents an essential singularity as $\rho \rightarrow 0$, while the second function $P(\rho)$ presents also a minimum for a finite value of ρ . The general solution of the equation (39) possesses the same features as found by Garfinkle and Melvin [2], implying that for any choice of ν the metric is singular.

3.4 Interpretation as solution of $D = 10$ supergravity

As was shown in [22], the FS model can be obtained via dimensional reduction of the $N = 1$, $D = 10$ supergravity, which contains apart from gravity and the dilaton field an antisymmetric tensor field \hat{H}_{ABC} . As a result, any on-shell configuration $(g_{\mu\nu}, A_\mu^a, B_\mu^a, \phi)$ in the model (26), can be uplifted to become a solution of ten-dimensional equations of motion for the $D = 10$ supergravity. The details of the compactification on the group manifold $S^3 \times S^3$ are given in [22], so we shall not repeat them here. We adopt the index convention used in [22], i. e. greek and latin indices refer to the four-dimensional and internal six-dimensional ($S^3 \times S^3$) spaces, respectively.

In the Einstein frame, the ten dimensional solution reads

$$ds_{10}^2 = e^{-3\phi/2} g_{\mu\nu} dx^\mu dx^\nu + 2e^{-\phi/2} \left(\Theta^{(1)a} \Theta^{(1)a} + \Theta^{(2)a} \Theta^{(2)a} \right), \quad (41)$$

where $g_{\mu\nu} dx^\mu dx^\nu$ is the four dimensional line element, $(a, b, c = 1, 2, 3)$,

$$\Theta^{(1)a} = A^a + \frac{\epsilon^a}{g_1}, \quad \Theta^{(2)a} = B^a + \frac{\epsilon^a}{g_2}, \quad A^a = A_\mu^a dx^\mu, \quad B^a = B_\mu^a dx^\mu, \quad F_{\mu\nu}^{(1)a} = F_{\mu\nu}^a, \quad F_{\mu\nu}^{(2)a} = G_{\mu\nu}^a,$$

while ϵ^a are the invariant 1-forms on S^3

$$\epsilon^1 = \cos \psi d\theta + \sin \psi \sin \theta_1 d\Phi, \quad \epsilon^2 = -\sin \psi d\theta + \cos \psi \sin \theta_1 d\Phi, \quad \epsilon^3 = d\psi + \cos \theta d\Phi,$$

ψ, θ, Φ being the Euler angles on the three sphere. The $D = 10$ dilaton field is $\hat{\phi} = \phi/2$, while the nonvanishing components of the ten-dimensional antisymmetric tensor field H_{ABC} are given by

$$H_{\alpha\beta a} = -\frac{1}{\sqrt{2}} e^{-3\phi/4} F_{\alpha\beta}^a, \quad H_{abc} = \frac{1}{\sqrt{2}} e^{3\phi/4} f_{abc}, \quad (42)$$

where $f_{abc} = f_{abc}^{(s)} = g_s \epsilon_{abc}$ ($s = 1, 2$) are the $SU(2)$ gauge group structure constants. Using the rules of [22], one can further lift the solutions to eleven dimensions to regard them in the context of M-theory.

3.5 On the supersymmetry of Melvin solution in FS model

We now want to study if the solution (27) preserve any amount of supersymmetry. For the rest of this section we use conventions similar to [21] and [23], in particular a mostly minus signature (*i.e.* we take $g_{\mu\nu} \rightarrow -g_{\mu\nu}$; the scalar and electromagnetic field preserve the form given above). We use $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$, $\sigma_{\alpha\beta} = \frac{1}{4}[\gamma_\alpha, \gamma_\beta]$, $\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, so that $\gamma_5^2 = 1$.

One of the supersymmetry transformations for a purely bosonic background and a vanishing axion read

$$\delta\bar{\chi} = \bar{\epsilon} \left(\frac{i}{\sqrt{2}} (\partial_\mu \phi) \gamma^\mu - \frac{1}{2} e^{-\phi} (\alpha^a F_{\mu\nu}^a + i\gamma_5 \beta^a G_{\mu\nu}^a) \sigma^{\mu\nu} + \frac{1}{4} e^\phi (g_1 + i\gamma_5 g_2) \right), \quad (43)$$

where $\epsilon = \epsilon^I$ are four Majorana spinors. Here α^a and β^a denote the generators of the $(1/2, 1/2)$ representation of $SU(2) \times SU(2)$. We use the form of these matrices given in [11], in particular

$$\alpha^3 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2 \end{pmatrix}, \quad \beta^3 = \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix}. \quad (44)$$

The condition that the variation of the Majorana field χ be vanishing can be written as

$$\bar{\epsilon}(\mathcal{M}_1 + \alpha^3 \mathcal{M}_2 + \beta^3 \mathcal{M}_3) = 0, \quad (45)$$

where the \mathcal{M}_i are 4×4 matrices given by

$$\begin{aligned} \mathcal{M}_1 &= \frac{i\Lambda'}{(2\Lambda)^{3/2}}\gamma_1 + \frac{1}{4}\sqrt{\frac{2c^2}{\Lambda}}(g_1 + i\gamma_5 g_2), \\ \mathcal{M}_2 &= -\frac{cB_0\alpha \cos u}{(2\Lambda)^{3/2}}\gamma_1\gamma_3, \\ \mathcal{M}_3 &= \frac{cB_0\alpha \sin u}{(2\Lambda)^{3/2}}i\gamma_5\gamma_1\gamma_3. \end{aligned} \quad (46)$$

which implies

$$\bar{\epsilon}\Theta = 0,$$

the 16×16 matrix Θ being defined as

$$\Theta = \begin{pmatrix} \Theta_+ & 0 \\ 0 & \Theta_- \end{pmatrix}, \quad \Theta_{\pm} = \begin{pmatrix} \mathcal{M}_1 & \mathcal{M}_2 \pm \mathcal{M}_3 \\ -(\mathcal{M}_2 \pm \mathcal{M}_3) & \mathcal{M}_1 \end{pmatrix}.$$

The necessary condition for the existence of Killing spinors $\bar{\epsilon}$ is thus $\det \Theta = 0$ which yields

$$B_0^2(g_2 \cos u \pm g_1 \sin u)^2 + \frac{\lambda^2}{4}(B_0^2 - 1)^2 = 0. \quad (47)$$

Therefore the conditions $g_2 \cos u = \pm g_1 \sin u$ and $B_0^2 = 1$ should be satisfied by the supersymmetric solutions. However, we prove that these conditions are too strong in order to allow for Killing spinors to exist. Returning to the equation (45), we write it in the form

$$\frac{\Lambda'}{c} + \alpha B_0 \hat{O}_1 + \lambda \Lambda \hat{O}_2 = 0 \quad (48)$$

where

$$\hat{O}_1 = i\alpha^3(\cos u - i\gamma_5 \sin u)\gamma_3, \quad \hat{O}_2 = \frac{i}{\lambda}(g_1 + i\gamma_5 g_2)\gamma_1 \quad (49)$$

satisfy the obvious relations $\hat{O}_1^2 = \hat{O}_2^2 = I$, $\{\hat{O}_1, \hat{O}_2\} = -2(g_1 \cos u - g_2 \sin u)/\lambda I$ (where I is the 4×4 unit matrix. One can easily show that (48) implies

$$\frac{\Lambda'^2}{c^2} = (B_0\alpha \cos u - g_1\Lambda)^2 + (B_0\alpha \sin u + g_2\Lambda)^2 \quad (50)$$

which is not compatible with the constraint (47). We conclude that, similar to the $a = 0$ Melvin solution [6], the configuration (27) does not preserve any supersymmetry.

4 Conclusions

Among configurations of fields that are in static equilibrium under their own gravitational attraction one of the simplest is a parallel bundle of magnetic flux. The corresponding configuration, known as Melvin solution, have found recently considerable attention in string/M theory context.

In this paper we have generalized the Melvin solution by including a Liouville-type dilaton potential in the action principle. Although it was not possible to solve the field equations in the general case, we have presented explicit solutions for several values of the theory constants. The Melvin solution in $D = 4$, $N = 4$ gauged supergravity has been extensively studied, and have many similar properties with the known solutions without a dilaton potential. In particular, we haven't found any fraction of supersymmetry being left unbroken for this configuration.

We did not considered the stability of our solutions against small radial, time-dependent perturbations. However, since the pure Melvin solution (which is recovered in the limit of vanishing dilaton potential) was shown to be stable [27, 29], one may expect that the FS configuration (27) to be also stable. However, it could decay by instanton processes similar to the case of vanishing dilaton potential.

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